Local Bases for Model-reduced Smoke Simulations
Supplementary Material

S1 Basis Construction in 3D

This section completes the basis construction method of Section 4.6. As explained in that section, the 3D basis flows used in the paper can be derived in a very simple manner from the 2D basis flows directly. However, if the basis flows are to be used for other applications, different constraints could be imposed on the basis flows, and the simple construction of Section 4.6 might not work. The method described here is more general, and also justifies the weighting in z given in Section 4.6.

Eigenflows aligned along the z axis are of the form

$$\mathbf{e}_z^{\boldsymbol{k}}(\mathbf{x}) = \begin{pmatrix} -k_x\, k_z\, \sin(k_x\, \pi\, x)\, \cos(k_y\, \pi\, y)\, \cos(k_z\, \pi\, z) \\ -k_y\, k_z\, \cos(k_x\, \pi\, x)\, \sin(k_y\, \pi\, y)\, \cos(k_z\, \pi\, z) \\ (k_x^2 + k_y^2)\, \cos(k_x\, \pi\, x)\, \cos(k_y\, \pi\, y)\, \sin(k_z\, \pi\, z) \end{pmatrix} \in \mathbb{R}^3\,, \tag{S1}$$

and $\mathbf{e}_x^{\boldsymbol{k}}$ and $\mathbf{e}_y^{\boldsymbol{k}}$ can be obtained by rotation. Note that even if we construct z-aligned templates, we combine together eigenflows aligned in x, y and z, using coefficients $w_{x,\mathbf{a}}^{\mathbf{k}}$, $w_{y,\mathbf{a}}^{\mathbf{k}}$, and $w_{z,\mathbf{a}}^{\mathbf{k}}$, respectively.

The 3D constraint system is larger than in 2D, but it has the same structure, and can be solved in the same way. Equations 9 become a set of six constaints (one for each face of the 3D basis flow template support). For the orthogonality constraints, we do not impose orthogonality between a basis flow and each of its neighbors of the same frequency, as this would create too many constraints. We instead only impose orthogonality among basis flows of the same axis alignment and same frequency located on the same axis-aligned slice. For instance, we impose that each z-aligned basis flow is orthogonal to each other z-aligned basis flow of the same frequency centered at the same z coordinate. This creates individual planes of orthogonal basis flows, which is sufficient for our use, as described in Section 5. These orthogonality constraints are therefore similar to the 2D case, and also reduce to three quadratic equations.

While using octave 0 is invalid in 2D since it creates an identically zero flow, it is a valid octave in 3D. It however creates problematic cases where eigenflows with different parameters are not linearly independent. For instance,

$$k_y\, \mathbf{e}_y^{(kx,ky,0)} = -k_x\, \mathbf{e}_x^{(k_x,k_y,0)}. \tag{S2}$$

These eigenflows cannot be avoided, since they portray the desired behavior for basis template $\mathbf{b}_z^{\mathbf{k}}$, i.e., a swirling motion around the z axis, at the scale of the basis template support. Equation 8 is therefore replaced in 3D by

$$k_y\, w_{y,(1,1,0)}^{\boldsymbol{k}} - k_x\, w_{x,(1,1,0)}^{\boldsymbol{k}} = 1. \tag{S3}$$

To remove linearly dependent eigenflows, we first look at each pair of eigenflows and remove one of them if one is a scaled version of the other. Then, after the linear constraints of the systems are resolved, we look at each remaining free coefficient, and compare the solutions obtained by setting the coefficient to 1 or 0. If both solutions are the same, the coefficient represents a null subspace, and can be discarded.

The octave set in 3D must be larger to account for the additional constraints, and we have found $\mathcal{A}_\star = \{0, 1, 2, 3, 5\}$ to be the smallest set to yield solutions. It however still gives us too many free variables, so we additionally impose that the basis templates be symmetric in the z direction with respect to their center. This reduces to linear constraints on the weights similar to Equations 9, and brings the number of free coefficients down to three. Finally, as in 2D, the three orthogonality constraints can be solved with exactly eight solutions, and we keep the real solution minimizing $\|\mathbf{h}^{\boldsymbol{k}}\|$.

Because of all the removed linearly dependent eigenflows, only 30 of the 375 coefficients are non-zero. They are given in Table S1 of this document for the most useful anisotropy ratios. Figure 5-left shows the structure of the 3D basis flows: z-aligned basis flows have no z component, and their xy-cuts have the same structure as the basis flows we constructed in 2D.

Direct computations can confirm numerically that this linear combination is the same as the basis template defined in Equation 16.

axis, $\boldsymbol{a}$	Anisotropy Ratio		
	(1 : 1 : 1)	(2 : 2 : 1)	(1 : 1 : 2)
y,(1,1,0)	1	0.5	1
y,(1,1,2)	-0.33333333	-0.33333333	-0.11111111
z,(1,1,2)	-0.33333333	-0.16666666	-0.22222222
y,(1,3,0)	-0.11072314	-0.05536157	-0.11072314
y,(1,3,2)	0.07908796	0.05032870	0.04258582
z,(1,3,2)	0.04745277	0.01509861	0.05110299
y,(1,5,0)	-0.13356611	-0.06678305	-0.13356611
y,(1,5,2)	0.11575729	0.06430960	0.08268378
z,(1,5,2)	0.04452203	0.01236723	0.06360291
y,(3,1,0)	-0.03690771	-0.01845385	-0.03690771
y,(3,1,2)	0.02636265	0.01677623	0.01419527
z,(3,1,2)	0.00527253	0.00167762	0.00567811
y,(3,3,0)	0.04209225	0.02104612	0.04209225
y,(3,3,2)	-0.03443911	-0.01993843	-0.02228413
z,(3,3,2)	-0.01147970	-0.00332307	-0.01485608
y,(3,5,0)	-0.01787380	-0.00893690	-0.01787380
y,(3,5,2)	0.01599235	0.00868156	0.01215419
z,(3,5,2)	0.00470363	0.00127670	0.00714952
y,(5,1,0)	-0.02671322	-0.01335661	-0.02671322
x,(5,1,2)	1.41909610	1.51847588	0.40189539
y,(5,1,2)	0.30697067	0.31655709	0.09691583
z,(5,1,2)	0.56941932	0.30418986	0.32406043
y,(5,3,0)	-0.01072428	-0.00536214	-0.01072428
x,(5,3,2)	-0.09040709	-0.05933200	-0.04599034
y,(5,3,2)	-0.04464884	-0.03039026	-0.02030169
z,(5,3,2)	-0.03446953	-0.01140678	-0.03421844
y,(5,5,0)	0.01177721	0.00588860	0.01177721
x,(5,5,2)	-0.12084482	-0.06721802	-0.08591584
y,(5,5,2)	-0.13174965	-0.07299117	-0.09483797
z,(5,5,2)	-0.05051889	-0.01402092	-0.07230152
$\|\mathbf{h}^{\widehat{\mathbf{k}}}\|$	1.19824688	1.19824688	0.84728849

Table S1: *Non-zero coefficients of eigenflows to construct our 3D basis templates.*